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Random times with a given Azéma supermartingale

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Begin at the beginning, and go on till you come to the end. Then, ...

Lewis Carroll, Alice's Adventures in Wonderland

Problem

Motivation: In credit risk, in mathematical finance, one works with random times which represent the default time. Many studies are based on the intensity process: starting with a reference filtration $\mathbb{F}$, the intensity process of τ is the $\mathbb{F}$ predictable increasing process Λ such that

$$\mathbb{1}_{\tau \leq t} - \Lambda_{t \wedge \tau}$$

is a $\mathbb{G}$-martingale, where $\mathcal{G}_t = \cap_{\epsilon > 0} \mathcal{F}_{t+\epsilon} \vee \sigma(\tau \wedge (t + \epsilon))$.

Then, the problem is : given Λ, construct a random time τ which admits Λ as intensity.

A classical construction is: assuming that the $\mathbb{F}$ adapted increasing process Λ is continuous, extend the probability space $(\Omega, \mathbb{F}, \mathbb{P})$ so that there exists a random variable Θ, with exponential law, independent of $\mathcal{F}_\infty$ and define
$\tau := \inf\{t : \Lambda_t \geq \Theta\}$.

Then,

$$\mathbb{P}(\tau > t|\mathcal{F}_t) = \mathbb{P}(\Lambda_t < \Theta|\mathcal{F}_t) = e^{-\Lambda_t}$$

Noting that any $\mathcal{G}_t$ measurable random variable Y_t satisfies

$$Y_t \mathbb{1}_{\{t<\tau\}} = y_t \mathbb{1}_{\{t<\tau\}}$$

where y_t is $\mathcal{F}_t$ adapted, it follows that, for X any integrable $\mathcal{F}_T$-measurable r.v.,

$$\mathbb{E}(X \mathbb{1}_{\{T<\tau\}}|\mathcal{G}_t) = \mathbb{1}_{t<\tau}\mathbb{E}(X e^{\Lambda_t - \Lambda_T}|\mathcal{F}_t)$$

A consequence is that $\mathbb{1}_{t<\tau}e^{\Lambda_t}$ is a $\mathbb{G}$ martingale. The martingale property of M is easily obtained.

Moreover, under this construction, one can show that any $\mathbb{F}$ martingale is a $\mathbb{G}$ martingale: this is the so-called immersion hypothesis.

Our goal is to provide other constructions. One starts with noting that, in general,

$$G_t = \mathbb{P}(\tau > t | \mathcal{F}_t)$$

is a supermartingale (called the Azema supermartingale) with multiplicative decomposition $G_t = N_t D_t$, where N is a local martingale and D a decreasing predictable process. Assuming that G does not vanishes, we set $D_t = e^{-\Lambda_t}$. Then, assuming that Λ is continuous, the Doob-Meyer decomposition of G is

$$dG_t = e^{-\Lambda_t} dN_t - Z_t d\Lambda_t = dm_t - dA_t$$

It follows that

$$\mathbb{1}_{\tau \le t} - \int_0^{t \wedge \tau} \frac{dA_s}{Z_s} = \mathbb{1}_{\tau \le t} - \int_0^{t \wedge \tau} d\Lambda_s$$

is a martingale, hence Λ is the intensity of τ.

Let $(\Omega, \mathbb{F}, \mathbb{P})$ be a filtered probability space, Z a supermartingale valued in $]0,1]$. Construct, on the canonical extended space $(\Omega \times [0,\infty])$, a probability $\mathbb{Q}$ and a random time τ (it will be the canonical map) such that

1. **restriction condition** $\mathbb{Q}|_{\mathcal{F}_\infty} = \mathbb{P}|_{\mathcal{F}_\infty}$ (we shall call $\mathbb{Q}$ an extension of $\mathbb{P}$)

2. **projection condition** $\mathbb{Q}[\tau > t|\mathcal{F}_t] = Z_t$

Using the multiplicative decomposition of a positive supermartingale, the problem can be written as:

Problem $(\star)$: let $(\Omega, \mathbb{F}, \mathbb{P})$ be a filtered probability space, Λ an increasing predictable process, N a non-negative local martingale such that

$$0 < N_t e^{-\Lambda_t} \leq 1$$

Construct, on the canonical extended space $(\Omega \times [0, \infty])$, a probability $\mathbb{Q}$ such that

1. **restriction condition** $\mathbb{Q}|_{\mathcal{F}_\infty} = \mathbb{P}|_{\mathcal{F}_\infty}$
2. **projection condition** $\mathbb{Q}[\tau > t|\mathcal{F}_t] = N_t e^{-\Lambda_t}$

We recall that τ is the canonical map. We shall note $\mathbb{P}(X) := \mathbb{E}_{\mathbb{P}}(X)$.

Föllmer's measure

One may think that a solution of the problem $(\star)$ is given by the Föllmer measure associated with Z, defined as

$$\mathbb{Q}^{\mathsf{F}}[F] = \mathbb{P}[\int_0^\infty F(s,\cdot)Z_s d\Lambda_s],\ F \in \mathcal{B}[0,\infty] \otimes \mathcal{F}_\infty.$$

which satisfies the projection condition.
In order to be a solution of the problem $(\star)$, $\mathbb{Q}^{\mathsf{F}}$ must be an extension of $\mathbb{P}$, i.e.,

$$\mathbb{P}[A] = \mathbb{Q}^{\mathsf{F}}[A] = \mathbb{P}[\mathbb{I}_A \int_0^\infty Z_s d\Lambda_s],\ A \in \mathcal{F}_\infty.$$

This is equivalent to the condition: $\int_0^\infty Z_s d\Lambda_s \equiv 1$. The last condition combined with the assumption $Z_\infty = 0$ implies, from the Doob-Meyer decomposition of Z written in differential form as $dZ_t = e^{-\Lambda_t} dN_t - Z_t d\Lambda_t$:

$$Z_t = \mathbb{P}[\int_0^\infty Z_s d\Lambda_s | \mathcal{F}_t] - \int_0^t Z_s d\Lambda_s = 1 - \int_0^t Z_s d\Lambda_s$$

i.e., $Z_t = e^{-\Lambda_t}$

Particular case: $Z = e^{-\Lambda}$.

In that case a solution (the Cox solution) is

$$\tau = \inf\{t : \Lambda_t \geq \Theta\}$$

where Θ is a random variable with unit exponential law, independent of $\mathcal{F}_\infty$, or in other words $\mathbb{Q} = \mathbb{Q}^C$ where, for $A \in \mathcal{F}_\infty$:

$$\mathbb{Q}^C(A \cap \{s < \tau \leq t\}) = \mathbb{P}\left(\mathbb{1}_A \int_s^t e^{-\Lambda_u} d\Lambda_u\right)$$

so that

$$\mathbb{Q}^C(\tau > \theta | \mathcal{F}_t) = e^{-\Lambda_\theta}, \text{ for } t \geq \theta$$

Jeanblanc, M. and Song, S. (2010)
Explicit Model of Default Time with given Survival Probability. *Stochastic Processes and their Applications*
Default times with given survival probability and their $\mathbb{F}$-martingale decomposition formula. *Stochastic Processes and their Applications*

Nikeghbali, A. and Yor, M. (2006) Doob's maximal identity, multiplicative decompositions and enlargements of filtrations, *Illinois Journal of Mathematics*, 50, 791-814. In that paper, given a supermartingale of the form $Z_t = \frac{N_t}{\sup_{s \leq t} N_s}$ where N is a continuous local martingale which goes to 0 at infinity, the authors show that $\mathbb{P}(g > t | \mathcal{F}_t) = Z_t$, where $g = \sup\{t \; : Z_t = 1\}$.

Li, L. and Rutkowski, M. (2010) Constructing Random Times Through Multiplicative Systems, Preprint.

In that paper, the authors give a solution to the problem $(\star)$, based on **Meyer, P.A. (1967)**: On the multiplicative decomposition of positive supermartingales. In: *Markov Processes and Potential Theory*, J. Chover, ed., J. Wiley, New York, pp. 103–116.

Outline of the talk

- Increasing families of martingales
- Semi-martingale decompositions
- Predictable Representation Theorem
- Exemple

The link between the supermartingale Z and the conditional law $\mathbb{Q}(\tau \in du|\mathcal{F}_t)$ for $u \le t$ is: Let $M_t^u = \mathbb{Q}(\tau \le u|\mathcal{F}_t)$, then M is increasing w.r.t. u and

$$\begin{aligned} M_u^u &= 1 - Z_u \\ M_t^u &\le M_t^t = 1 - Z_t \end{aligned}$$

(Note that, for $t < u$, $M_t^u = \mathbb{E}(1 - Z_u|\mathcal{F}_t)$).

The solution of problem $(\star)$ is not unique, mainly because the knowledge of the survival probability $\mathbb{Q}(\tau > t|\mathcal{F}_t)$ does not contain enough information to reconstruct the whole conditional law, i.e. $\mathbb{Q}(\tau \in du|\mathcal{F}_t)$.

Solving the problem $(\star)$ is equivalent to find a family M^u

Family iM_Z

An increasing family of positive martingales bounded by $1 - Z$ (in short iM_Z) is a family of processes $(M^u : 0 < u < \infty)$ satisfying the following conditions:

1. Each M^u is a càdlàg $\mathbb{P}$-$\mathbb{F}$ **martingale** on $[u, \infty]$.
2. For any u, the martingale M^u is positive and closed by $M^u_\infty = \lim_{t\to\infty} M^u_t$.
3. For each fixed t, $0 < t \leq \infty$, $u \in [0, t] \to M^u_t$ **is a right continuous increasing map**.
4. $M^u_u = 1 - Z_u$ and $M^u_t \leq M^t_t = 1 - Z_t$ for $u \leq t \leq \infty$.

Given an iM_Z, let $d_u M_\infty^u$ be the random measure on $(0, \infty)$ associated with the increasing map $u \to M_\infty^u$. The following probability measure $\mathbb{Q}$ is a solution of the problem $(\star)$

$$\mathbb{Q}(F) := \mathbb{P}\left(\int_{[0,\infty]} F(u, \cdot)\big(M_\infty^0 \delta_0(du) + d_u M_\infty^u + (1 - M_\infty^\infty)\delta_\infty(du)\big) \right)$$

The two properties for $\mathbb{Q}$:

- **Restriction condition**: For $B \in \mathcal{F}_\infty$,

$$\mathbb{Q}(B) = \mathbb{P}\left(\mathbb{I}_B \int_{[0,\infty]} (M_\infty^0 \delta_0(du) + d_u M_\infty^u + (1 - M_\infty^\infty)\delta_\infty(du)) \right) = \mathbb{P}[B]$$

- **Projection condition:** For $0 \le t < \infty$, $A \in \mathcal{F}_t$,

$$\mathbb{Q}[A \cap \{\tau \le t\}] = \mathbb{P}[\mathbb{I}_A M_\infty^t] = \mathbb{P}[\mathbb{I}_A M_t^t] = \mathbb{Q}[\mathbb{I}_A (1 - Z_t)]$$

are satisfied.

Notice that the advantage to consider an unknown iM_Z instead of an unknown $\mathbb{Q}$ is that iM_Z is a process which can be constructed on the initial space $(\Omega, \mathbb{F}, \mathbb{P})$, while $\mathbb{Q}$ is probability on an unknown space.

A solution of the $(\star)$-problem exists if and only if an iM_Z exists.

Constructions of iM_Z

Hypothesis (✠)

1. $Z_0 = 1$ and Λ is continuous.

2. For all $0 < t < \infty$, $0 \le Z_t < 1, 0 \le Z_{t-} < 1$ (strictly smaller than 1).

The simplest iM_Z

Assume conditions (✠). The family

$$M_t^u := (1 - Z_t) \exp\left(- \int_u^t \frac{Z_s}{1 - Z_s} d\Lambda_s \right) \ 0 < u < \infty, u \le t \le \infty,$$

defines an iM_Z, called **basic solution**. We note that

$$dM_t^u = -M_{t-}^u \frac{e^{-\Lambda_t}}{1 - Z_{t-}} dN_t, \ 0 < u \le t < \infty.$$

$M_t^u = \mathbb{Q}(\tau < u | \mathcal{F}_t), \quad Z_t = 1 - M_t^t = N_t e^{-\Lambda_t}$

It follows that, for $A \in \mathcal{B}([t,\infty[)$

$$\mathbb{Q}(\tau \in A|\mathcal{F}_t) = \int_A Z_s E_t(s) d\Lambda_s$$

where $E_t(s) := \frac{1-Z_t}{1-Z_s} \exp - \int_s^t \frac{Z_u}{1-Z_u} d\Lambda_u$.

Note that $M_t^u = m_t A_u$ where m is a martingale and A increasing.

Other solutions Let us recall that, to construct an iM_Z, we should respect four constraints :

i. $M^u_u = (1 - Z_u)$

ii. $0 \le M^u$

iii. $M^u \le 1 - Z$

iv. $M^u \le M^v$ for $u < v$

These constraints are particularly "easy" to handle if M^u are solutions of a SDE:
The constraint i indicates the initial condition;
the constraint ii means that we must take an exponential SDE;
the constraint iv is a comparison theorem for one dimensional SDE,
the constraint iii can be handled by local time as described in the following result :

Let m be a $(\mathbb{P}, \mathbb{F})$-local martingale such that $m_u \le 1 - Z_u$. Then, $m_t \le (1 - Z_t)$ on $t \in [u, \infty)$ if and only if the local time at zero of $m - (1 - Z)$ on $[u, \infty)$ is identically null.

Generating equation when $1 - Z > 0$

Hypothesis (✠✠):

1. $Z_0 = 1$ and Λ is continuous.
2. For all $0 < t < \infty$, $0 \leq Z_t < 1, 0 \leq Z_{t-} < 1$.
3. All $\mathbb{P}$-$\mathbb{F}$ martingales are continuous.

Assume (✠✠). Let Y be a $(\mathbb{P}, \mathbb{F})$ local martingale and f be a bounded Lipschitz function with $f(0) = 0$. For any $0 \leq u < \infty$, we consider the equation

$$(\natural_u) \left\{ \begin{array}{rcl} dX_t &=& X_t \left(-\frac{e^{-\Lambda_t}}{1 - Z_t} dN_t + f(X_t - (1 - Z_t)) dY_t \right), \ u \leq t < \infty \\ X_u &=& x \end{array} \right.$$

Generating equation when $1-Z>0$

Hypothesis (✠✠):

1. $Z_0 = 1$ and Λ is continuous.
2. For all $0<t<\infty$, $0\leq Z_t<1, 0\leq Z_{t-}<1$ (strictly smaller than 1).
3. All $\mathbb{P}$-$\mathbb{F}$ martingales are continuous.

Assume (✠✠). Let Y be a $(\mathbb{P},\mathbb{F})$ local martingale and f be a bounded Lipschitz function with $f(0)=0$. For any $0\leq u<\infty$, we consider the equation

$$(\natural_u)\left\{\begin{array}{rcl} dX_t &=& X_t\left(-\frac{e^{-\Lambda_t}}{1-Z_t}dN_t+f(X_t-(1-Z_t))dY_t\right),\ u\leq t<\infty \\ X_u &=& x \end{array}\right.$$

Let M^u be the solution on $[u,\infty)$ of the equation $(\natural_u)$ with initial condition $M^u_u = 1-Z_u$. Then, $(M^u, u\leq t<\infty)$ defines an iM_Z.

A remark

Our method remains valid if in SDE($\natural$)
$dM_t = M_t\left(-\frac{e^{-\Lambda_t}}{1-Z_t}dN_t + f(M_t - (1-Z_t))dY_t\right)$, the term $f(M_t - (1-Z_t))$ is replaced by some more general function $f(M_t - (1-Z_t), M_t, t, \omega)$ such that

$$|f(M_t - (1-Z_t), M_t, t, \omega)| \leq K|M_t - (1-Z_t)|$$

.

Proof

• Inequality $M^u \leq 1 - Z$ on $[u, \infty)$ is satisfied if the local time of $\Delta = M^u - (1 - Z)$ at zero is null. This is the consequence of the following estimation:

$$\begin{aligned} d\langle\Delta\rangle_t &= \Delta_t^2 \left(\frac{e^{-\Lambda_t}}{1 - Z_t}\right)^2 d\langle N\rangle_t + M_t^2 f^2(\Delta_t) d\langle Y\rangle_t - 2\Delta_t \frac{e^{-\Lambda_t}}{1 - Z_t} M_t f(\Delta_t) d\langle N, Y\rangle_t \\ &\leq 2\Delta_t^2 \left(\frac{e^{-\Lambda_t}}{1 - Z_t}\right)^2 d\langle N\rangle_t + 2M_t^2 f^2(\Delta_t) d\langle Y\rangle_t \\ &\leq 2\Delta_t^2 \left(\frac{e^{-\Lambda_t}}{1 - Z_t}\right)^2 d\langle N\rangle_t + 2M_t^2 K^2 \Delta_t^2 d\langle Y\rangle_t \end{aligned}$$

From this, we can write

$$\int_0^t \mathbb{I}_{\{0<\Delta_s<\epsilon\}} \frac{1}{\Delta_s^2} d\langle\Delta\rangle_s < \infty, \ 0 < \epsilon, 0 < t < \infty$$

and get the result according to Revuz-Yor.

• Inequality $M^u \leq M^v$ on $[v,\infty)$ when $u<v$. The comparison theorem holds for SDE($\natural$). We note also that M^u and M^v satisfy the same SDE($\natural$) on $[v,\infty)$. So, since $M^u_v \leq (1-Z_v) = M^v_v$, $M^u_t \leq M^v_t$ for all $t \in [v,\infty)$.

Comments: in the case $N = 1$, one obtains

$$\begin{cases} dX_t &= X_t \left(f(X_t - (1 - Z_t)) dY_t \right), \ u \leq t < \infty \\ X_u &= x \end{cases}$$

In that case, the Azema supermartingale is decreasing. This is a caracterization of pseudo-stopping times (i.e. times such that, for any BOUNDED $\mathbb{F}$ martingale m, one has

$$\mathbb{E}(m_\tau) = m_0$$

Balayage formula when $1 - Z$ can reach zero

We introduce $\mathcal{Z} = \{s : 1 - Z_s = 0\}$ and, for $t \in (0, \infty)$, the random time

$$g_t \quad := \quad \sup\{0 \leq s \leq t : s \in \mathcal{Z}\}$$

Hypothesis($\mathcal{Z}$) The set $\mathcal{Z}$ is not empty and is closed.
The measure $d\Lambda$ has a decomposition $d\Lambda_s = dV_s + dA_s$ where V, A are continuous increasing processes such that dV charges only $\mathcal{Z}$ while dA charges its complementary $\mathcal{Z}^c$.
Moreover, we suppose

$$\mathbb{I}_{\{g_t \leq u < t\}} \int_u^t \frac{Z_s}{1 - Z_s} dA_s < \infty$$

for any $0 < u < t < \infty$.

We suppose that Λ is continuous, $Z_0 = 1$ and $\mathbf{Hy}(\mathcal{Z})$. We introduce

$$M_t^u \;=\; \mathbb{I}_{\{g_t \leq u\}} \exp\left(-\int_u^t \frac{Z_s}{1-Z_s} dA_s\right)(1-Z_t),\;\; 0<u<\infty, u\leq t\leq \infty.$$

The family $(M^u : 0 \leq u < \infty)$ defines an iM_Z. Moreover, for $0 < u \leq t \leq \infty$

$$M_t^u = (1-Z_u) - \int_u^t \mathbb{I}_{\{g_s \leq u\}} \exp\left(-\int_u^s \frac{Z_v}{1-Z_v} dA_v\right) e^{-\Lambda_s} dN_s$$

Proof indication

(Balayage Formula.) Let Y be a continuous semi-martingale and define

$$g_t = \sup\{s \leq t : Y_s = 0\},$$

with the convention $\sup\{\emptyset\} = 0$. Then

$$h_{g_t} Y_t = h_0 Y_0 + \int_0^t h_{g_s} dY_s$$

for every predictable, locally bounded process h.

We need only to prove that each M^u satisfies the above equation, and therefore, that M^u is a local $\mathbb{P}$-$\mathbb{F}$ martingale. Let

$$E_t^u = \exp\left(-\int_u^t \frac{Z_s}{1-Z_s} dA_s\right)$$

Then,

$$d\left(E_t^u(1-Z_t)\right) = E_t^u\left(-e^{-\Lambda_t} dN_t + Z_t dV_t\right)$$

We apply the balayage formula and we obtain

$$\begin{aligned}
M_t^u &= \mathbb{I}_{\{g_t \le u\}} E_t^u (1-Z_t) \\
&= \mathbb{I}_{\{g_t \le u\}} (1-Z_u) + \int_u^t \mathbb{I}_{\{g_s \le u\}} E_s^u \left(-e^{-\Lambda_s} dN_s + Z_s dV_s\right) \\
&= (1-Z_u) - \int_u^t \mathbb{I}_{\{g_s \le u\}} E_s^u e^{-\Lambda_s} dN_s
\end{aligned}$$

Enlargement of filtration problem solved by SDE

Here we study in particular the enlargement of filtration problem.

- $\mathbb{G}$ is a progressive enlargement of $\mathbb{F}$.
- The $\mathbb{F}$-local martingales remain always $\mathbb{G}$-semimartingales on the interval $[0, \tau]$. whose semimartingale decomposition formula is given in Jeulin.
- The $\mathbb{F}$-local martingales' behaviour on the interval $[\tau, \infty)$ in the filtration $\mathbb{G}$ depends on the model.

Semimartingale decomposition formula for the models constructed with SDE($\natural$), in the case $1 - Z > 0$

We suppose

• **Hy**(✠✠) and $Z_\infty = 0$

• for each $0 \le t \le \infty$, the map $u \to M_t^u$ is continuous on $[0, t]$, where M^u is solution of the generating equation ($\natural$): $0 \le u < \infty$,

$$(\natural_u) \begin{cases} dM_t &= M_t \left(-\frac{e^{-\Lambda_t}}{1-Z_t} dN_t + f(M_t - (1 - Z_t)) dY_t \right), \ u \le t < \infty \\ M_u &= 1 - Z_u \end{cases}$$

We prove that, for our models, the hypothesis $(\mathcal{H}')$ holds between $\mathbb{F}$ and $\mathbb{G}$ and we obtain semimartingale decomposition formula.

Let $\mathbb{Q}$ be the probability on the product space $[0,\infty]\otimes\Omega$ associated with the iM_Z

Let X be a $\mathbb{P}$-$\mathbb{F}$ local martingale. Then the process

$$\begin{aligned}\widetilde{X}_t &= X_t - \int_0^t 1\!\!1_{\{s\le\tau\}}\frac{e^{-\Lambda_s}}{Z_s}d\langle N,X\rangle_s + \int_0^t 1\!\!1_{\{\tau<s\}}\frac{e^{-\Lambda_s}}{1-Z_s}d\langle N,X\rangle_s \\ &\quad - \int_0^t 1\!\!1_{\{\tau<s\}}(f(M_s^\tau-(1-Z_s))+M_s^\tau f'(M_s^\tau-(1-Z_s)))d\langle Y,X\rangle_s\end{aligned}$$

is a $\mathbb{Q}$-$\mathbb{G}$-local martingale.

Sketch of the proof:

We compute directly the expectations

$$\mathbb{Q}[1\!\!1_A 1\!\!1_{\{\tau\le u\}}(X_{\tau\wedge t}-X_{\tau\wedge s})], A\in\mathcal{F}_s, 0\le u<\infty.$$

Let $0\le a<b\le s<t$ and $A\in\mathcal{F}_s$.

$$\begin{aligned}
& \mathbb{Q}[1\!\!1_A 1\!\!1_{\{a<\tau\le b\}}(X_t-X_s)] \\
= \;& \mathbb{Q}[1\!\!1_A (M^b_\infty - M^a_\infty)(X_t-X_s)] \\
= \;& \mathbb{Q}[1\!\!1_A \int_s^t \frac{(-1)e^{-\Lambda_r}}{1-Z_r}(M^b_r-M^a_r)\,d\langle N,X\rangle_r] \\
+ \;& \mathbb{Q}[1\!\!1_A \int_s^t \left(M^b_r f(M^b_r-(1-Z_r)) - M^a_r f(M^a_r-(1-Z_r))\right) d\langle Y,X\rangle_r]
\end{aligned}$$

Compute separately the two terms in the right-hand side. Firstly

$$
\begin{aligned}
&\mathbb{Q}[1\!\!1_A \int_s^t \big(M_r^b f(M_r^b - (1-Z_r))\big)\, d\langle Y, X\rangle_r] \\
= \quad & \mathbb{Q}[1\!\!1_A \int_s^t \frac{(-1)e^{-\Lambda_r}}{1-Z_r}(M_\infty^b - M_\infty^a)\, d\langle N, X\rangle_r] \\
= \quad & \mathbb{Q}[1\!\!1_A 1\!\!1_{\{a<\tau\le b\}} \int_s^t (-\frac{e^{-\Lambda_r}}{1-Z_r}) d\langle N, X\rangle_r]
\end{aligned}
$$

For the second term

$$
\begin{aligned}
&\mathbb{Q}[1\!\!1_A \int_s^t \big(M_r^b f(M_r^b - (1-Z_r)) - M_r^a f(M_r^a - (1-Z_r))\big)\, d\langle Y, X\rangle_r] \\
= \quad & \mathbb{Q}[1\!\!1_A \int_s^t \int_a^b (f(M_r^v - (1-Z_r)) + M_r^v f'(M_r^v - (1-Z_r))) d_v M_r^v d\langle Y, X\rangle_r] \\
= \quad & \mathbb{Q}[1\!\!1_A \int_s^t 1\!\!1_{\{a<\tau\le b\}} (f(M_r^\tau - (1-Z_r)) + M_r^\tau f'(M_r^\tau - (1-Z_r))) d\langle Y, X\rangle_r]
\end{aligned}
$$

It is now easy to deduce the semimartingale decomposition formula from this computation.

Semimartingale decomposition formula for the model constructed with the balayage formula, the case of eventual $1 - Z = 0$

We suppose that Λ is continuous, $Z_0 = 1$ and $\mathbf{Hy}(\mathcal{Z})$. We consider the iM_Z constructed above and its associated probability measure $\mathbb{Q}$ on $[0, \infty] \times \Omega$. Let $g = lim_{t\to\infty} g_t$.

Let X be a $(\mathbb{P}, \mathbb{F})$-local martingale. Then

$$X_t - \int_0^t \mathbb{1}_{\{s\leq g\vee\tau\}} \frac{e^{-\Lambda_s}}{Z_{s-}} d\langle N, X\rangle_s + \int_0^t \mathbb{1}_{\{g\vee\tau<s\}} \frac{e^{-\Lambda_s}\ d\langle N, X\rangle_s}{1 - Z_{s-}},\ 0 \leq t < \infty,$$

is a $(\mathbb{Q}, \mathbb{G})$-local martingale.

It is noted that the above formula has the same form as the formula for honest time, whilst $g \vee \tau$ is not a honest time in the filtration $\mathbb{F}$.

Proof with the SDE

The theorem can be proved in quite the same way as in the preceding theorem, except some precaution on the zeros of $1-Z$. Recall that the elements in iM_Z satisfy the equation:

$$M_t^u = (1-Z_u) - \int_u^t \mathbb{I}_{\{g_s \le u\}} E_s^u e^{-\Lambda_s} dN_s, \ u \le t < \infty.$$

Let $0 \le a < b \le s < t$ and $A \in \mathcal{F}_s$. Put aside the integrability question. We have

$$\begin{array}{rl} & \mathbb{Q}[1\!\!1_A 1\!\!1_{\{a<g\vee\tau\le b\}}(X_t - X_s)] = \mathbb{Q}[1\!\!1_A (M_\infty^b - M_\infty^a)(X_t - X_s)] \\ = & \mathbb{Q}[1\!\!1_A \int_s^t 1\!\!1_{\{g_r \le b\}} E_r^b (-e^{-\Lambda_r}) d\langle N, X\rangle_r] - \mathbb{Q}[1\!\!1_A \int_s^t 1\!\!1_{\{g_r \le a\}} E_r^a (-e^{-\Lambda_r}) d\langle N, X\rangle_r] \\ = & \mathbb{Q}[1\!\!1_A 1\!\!1_{\{g\vee\tau\le b\}} \int_s^t \frac{(-e^{-\Lambda_r})}{1-Z_r} d\langle N, X\rangle_r] - \mathbb{Q}[1\!\!1_A 1\!\!1_{\{g\vee\tau\le a\}} \int_s^t \frac{(-e^{-\Lambda_r})}{1-Z_r} d\langle N, X\rangle_r] \\ = & \mathbb{Q}[1\!\!1_A 1\!\!1_{\{a<g\vee\tau\le b\}} \int_s^t \frac{(-e^{-\Lambda_r})}{1-Z_r} d\langle N, X\rangle_r] \end{array}$$

Predictable Representation Property

Assume ✠✠ and that

1. there exists an $(\mathbb{P}, \mathbb{F})$-martingale m which admits the $(\mathbb{P}, \mathbb{F})$-Predictable Representation Property
2. The martingales N and Y are orthogonal

Let $\tilde{m}$ be the $(\mathbb{P}, \mathbb{G})$-martingale part of the $(\mathbb{P}, \mathbb{G})$-semimartingale m.
Then, $(\tilde{m}, M)$ enjoys the $(\mathbb{Q}, \mathbb{G})$-Predictable Representation Property where $M_t = \mathbb{1}_{\tau \leq t} - \Lambda_{t \wedge \tau}$.

Example

Let φ is the standard Gaussian density and Φ the Gaussian cumulative function, $\mathbb{F}$ generated by a Brownian motion B.

Let $X = \int_0^\infty f(s)dB_s$ where f is a deterministic, square-integrable function and $Y = \psi(X)$ where ψ is a positive and strictly increasing function. Then,

$$\mathbb{P}(Y \le u|\mathcal{F}_t) = \mathbb{P}\Big(\int_t^\infty f(s)dB_s \le \psi^{-1}(u) - m_t|\mathcal{F}_t\Big)$$

where $m_t = \int_0^t f(s)dB_s$ is $\mathcal{F}_t$-measurable. It follows that

$$M_t^u := \mathbb{P}(Y \le u|\mathcal{F}_t) = \Phi\Big(\frac{\psi^{-1}(u) - m_t}{\sigma(t)}\Big)$$

The family M_t^u is then a family of iM_Z martingales which satisfies

$$dM_t^u = -\varphi\left(\Phi^{-1}(M_t^u)\right)\frac{f(t)}{\sigma(t)}dB_t$$

The multiplicative decomposition of $Z_t = N_t \exp\left(-\int_0^t \lambda_s ds\right)$ where

$$
\begin{aligned}
dN_t &= N_t \frac{\varphi(Y_t)}{\sigma(t)\Phi(Y_t)} dm_t, \quad \lambda_t = \frac{h'(t)\,\varphi(Y_t)}{\sigma(t)\,\Phi(Y_t)} \\
Y_t &= \frac{m_t - \psi^{-1}(t)}{\sigma(t)}
\end{aligned}
$$

The basic martingale satisfies

$$
dM_t^u = -M_t^u \frac{f(t)\varphi(Y_t)}{\sigma(t)\Phi(-Y_t)} dB_t.
$$

Open question: Is it possible to characterize supermartingales Z so that τ can be constructed on Ω

Gapeev, P. V., Jeanblanc, M., Li, L., and Rutkowski, M. (2009): Constructing Random Times with Given Survival Processes and Applications to Valuation of Credit Derivatives. Forthcoming in: *Contemporary Quantitative Finance* Springer-Verlag 2010.

In that paper, the probability $\mathbb{Q}$ is constructed as a probability measure equivalent to the solution of Cox model $\mathbb{Q}^C$ on $[0,\infty] \times \Omega$ associated with Λ. Define

$$d\mathbb{Q}|_{\mathcal{G}_t} = L_t d\mathbb{Q}^C|_{\mathcal{G}_t}, \ 0 \le t < \infty$$

where $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(\tau \wedge t)$ and $L_t = \ell_t 1\!\!1_{t<\tau} + L_t(\tau) 1\!\!1_{\tau \le t}$. If L satisfies

$$\begin{aligned} \ell_t &= N_t, \\ \texttt{(L)}: \quad N_t e^{-\Lambda_t} + \int_0^t L_t(s) e^{-\Lambda_s} d\Lambda_s &= 1, \ 0 \le t < \infty. \end{aligned}$$

where, for any s, the process $(L_t(s), t \ge s)$ **is an $\mathbb{F}$-martingale** satisfying $L_s(s) = N_s$, then, $\mathbb{Q}$ is a solution of the problem $(\star)$.

Conditions: **find** $L_t = \ell_t 1\!\!1_{t<\tau} + L_t(\tau, \cdot) 1\!\!1_{\tau \le t}$ **such that**

$$\begin{array}{rcl} \ell_t & = & N_t, \\ (\mathtt{L}): \quad N_t e^{-\Lambda_t} + \int_0^t L_t(s, \cdot) e^{-\Lambda_s} d\Lambda_s & = & 1, \; 0 \le t < \infty. \end{array}$$

where $(L_t(s), t \ge s)$ **is an** $\mathbb{F}$**-martingale satisfying** $L_s(s) = N_s$.

- The form of L is a general form for $\mathbb{G}$-adapted processes
- The condition on martingality of $L_t(s), t \ge s$ is to ensure that L is a $\mathbb{G}$-martingale
- The condition $L_t 1\!\!1_{t<\tau} = N_t 1\!\!1_{t<\tau}$ is stated to satisfy the **projection condition**
- The condition $(\mathtt{L})$ is needed to satisfy **the restriction condition** (and implies that L is a $\mathbb{G}$-martingale).

In fact, the process $L_t = \ell_t 1\!\!1_{t<\tau} + L_t(\tau, \cdot) 1\!\!1_{\tau \le t}$ is a $\mathbb{G}$ local martingale iff $L_t(s), t \ge s$ and $\mathbb{E}(L_t | \mathcal{F}_t)$ are $\mathbb{F}$-martingales.

To solve $(\mathtt{L})$ the idea is to find X and Y so that $L_t(s) = X_t Y_s$ and $N_t = X_t Y_t$.

Conditions: **find** $L_t = \ell_t 1\!\!1_{t<\tau} + L_t(\tau,\cdot) 1\!\!1_{\tau\le t}$ **such that**

$$\begin{aligned} \ell_t &= N_t, \\ (\mathtt{L}): \quad N_t e^{-\Lambda_t} + \int_0^t L_t(s,\cdot) e^{-\Lambda_s} d\Lambda_s &= 1,\ 0 \le t < \infty. \end{aligned}$$

where $(L_t(s), t \ge s)$ **is an** $\mathbb{F}$**-martingale satisfying** $L_s(s) = N_s$.

- The form of L is a general form for $\mathbb{G}$-adapted processes
- The condition on martingality of $L_t(s), t \ge s$ is to ensure that L is a $\mathbb{G}$-martingale
- The condition $L_t 1\!\!1_{t<\tau} = N_t 1\!\!1_{t<\tau}$ is stated to satisfy the **projection condition**
- The condition ($\mathtt{L}$) is needed to satisfy **the restriction condition** (and implies that L is a $\mathbb{G}$-martingale).

In fact, the process $L_t = \ell_t 1\!\!1_{t<\tau} + L_t(\tau,\cdot) 1\!\!1_{\tau\le t}$ is a $\mathbb{G}$ local martingale iff $\mathbb{E}(L_t|\mathcal{F}_t)$ and for any s, $L_t(s), t \ge s$ are $\mathbb{F}$-martingales.

To solve ($\mathtt{L}$) the idea is to find X and Y so that $L_t(s) = X_t Y_s$ and $N_t = X_t Y_t$.

Conditions: **find $L_t = \ell_t 1\!\!1_{t<\tau} + L_t(\tau,\cdot)1\!\!1_{\tau\le t}$ such that**

$$\begin{aligned} \ell_t &= N_t, \\ (\mathtt{L}): \quad N_t e^{-\Lambda_t} + \int_0^t L_t(s,\cdot)e^{-\Lambda_s} d\Lambda_s &= 1,\ 0 \le t < \infty. \end{aligned}$$

where $(L_t(s), t \ge s)$ is an $\mathbb{F}$-martingale satisfying $L_s(s) = N_s$.

- The form of L is a general form for $\mathbb{G}$-adapted processes
- The condition on martingality of $L_t(s), t \ge s$ is to ensure that L is a $\mathbb{G}$-martingale
- The condition $L_t 1\!\!1_{t<\tau} = N_t 1\!\!1_{t<\tau}$ is stated to satisfy the **projection condition**
- The condition $(\mathtt{L})$ is needed to satisfy **the restriction condition** (and implies that L is a $\mathbb{G}$-martingale).

In fact, the process $L_t = \ell_t 1\!\!1_{t<\tau} + L_t(\tau,\cdot)1\!\!1_{\tau\le t}$ is a $\mathbb{G}$ local martingale iff $\mathbb{E}(L_t|\mathcal{F}_t)$ and for any s, $L_t(s), t \ge s$ are $\mathbb{F}$-martingales.

To solve $(\mathtt{L})$ the idea is to find X and Y so that $L_t(s) = X_t Y_s$ and $N_t = X_t Y_t$.

Begin at the beginning, and go on till you come to the end. Then, stop.
Lewis Carroll, Alice's Adventures in Wonderland

This should be the end of the [talk], but not the end of research

St Augustin

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Thank you for your attention